Dynamic Time Series Model

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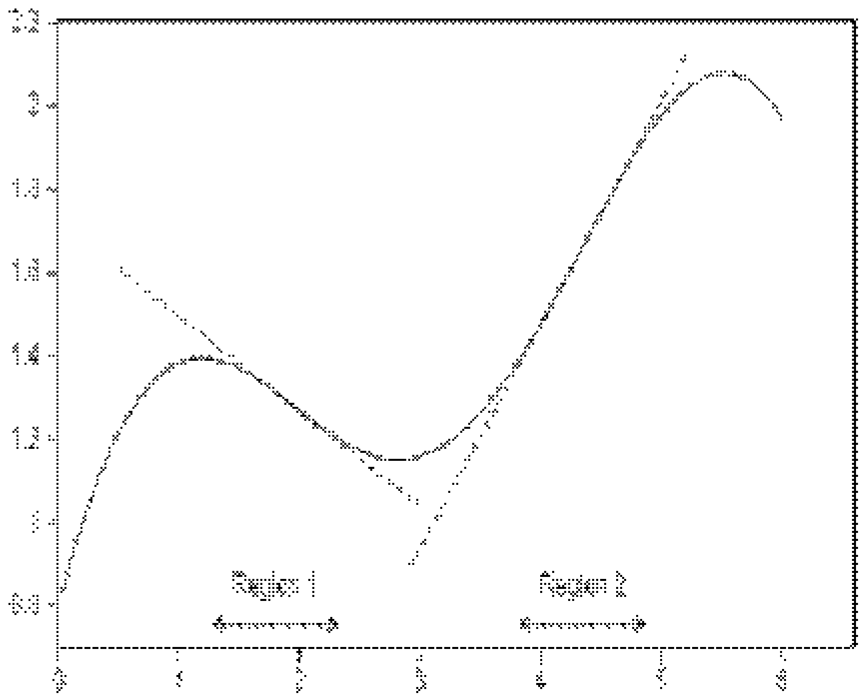
Dynamic Linear Model

$$y_t = x_t + \nu_t \qquad\qquad x_t = a_t + b_t X_t$$

$\mathbf{F}_t = (1, X_t)'$ and $\theta_t = (a_t, b_t)'$ "wanders" through time



FFBS Algorithm

Forward Filtering

- Normal linear analysis by Kalman filter
- Delivers normal $\mathbf{?}_{\mathrm{n}}$ $\text{from}\,(\mathbf{?}_{\mathrm{n}}\,|\mathrm{D}_{\mathrm{n}}) \sim \mathrm{N}(\mathbf{m}_{\mathrm{n}}, \mathbf{C}_{\mathrm{n}})$

Backward Sampling

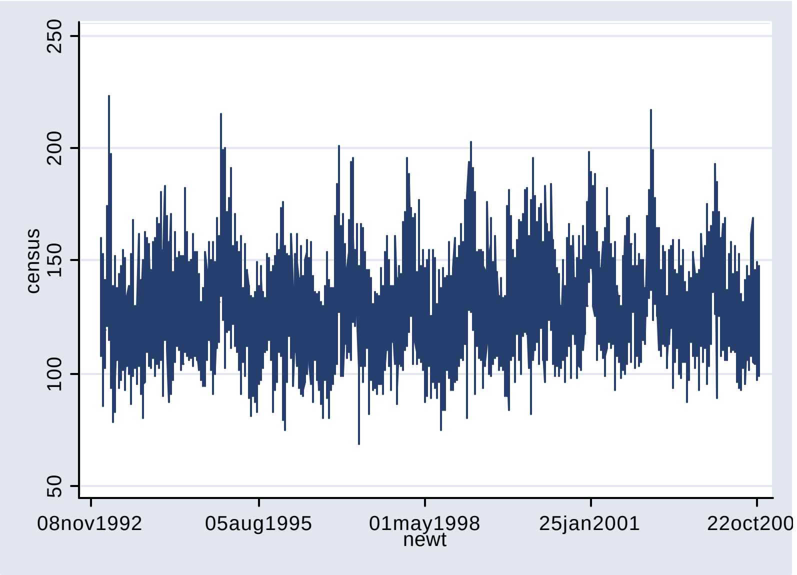
- At t=n, sample, $\mathbf{?}_t$ $\mathit{from}\,(\mathbf{?}_t\,|D_t) \sim N(\mathbf{m}_t, \mathbf{C}_t)$

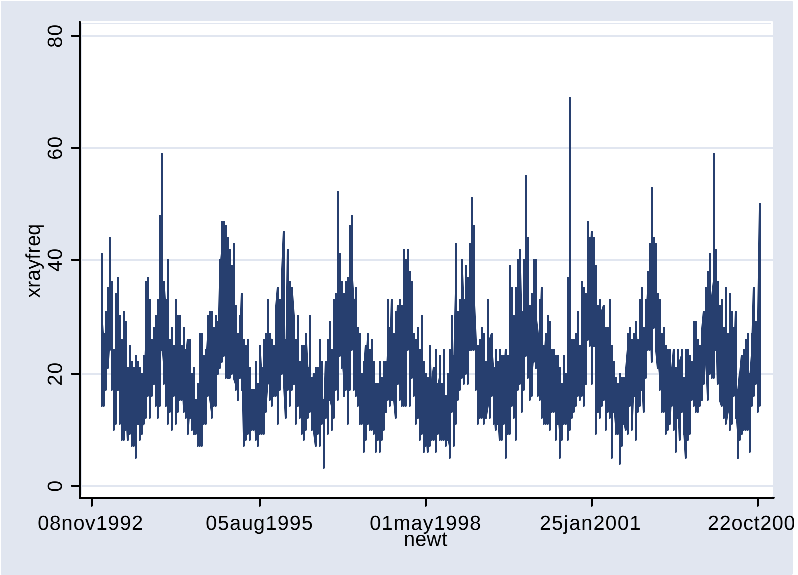
$$for\ t = n-1,\ n-2, \ldots, 1,\ sample\ \mathbf{?}_t^{*}\ \ from\ P(\mathbf{?}_t \mid D_t,\ \mathbf{?}_{t+1}^{*})$$

Dynamic Model on Time Series

- Data: Two time series data
- Number of people who come to ED a day from Jan.1. 1993 ~ Nov. 10. 2003
- Number of people who have got chest radiography
- Two sequences are correlated. Corr=0.3

Data Description





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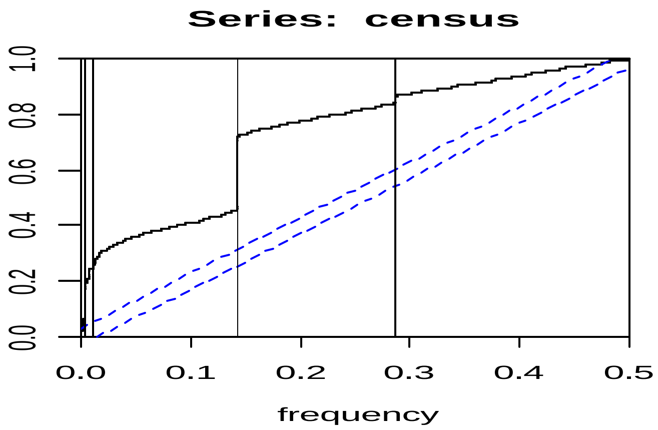
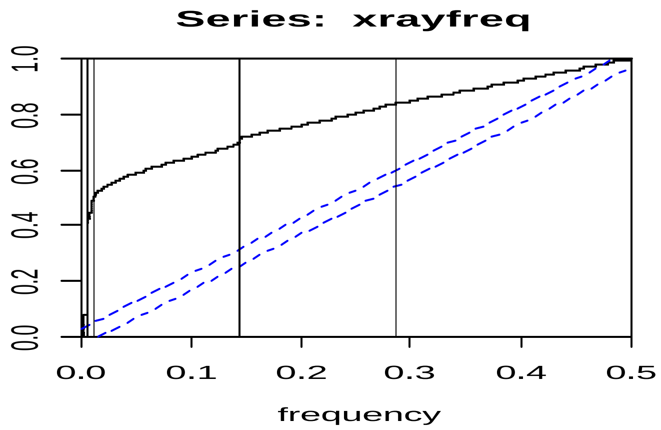
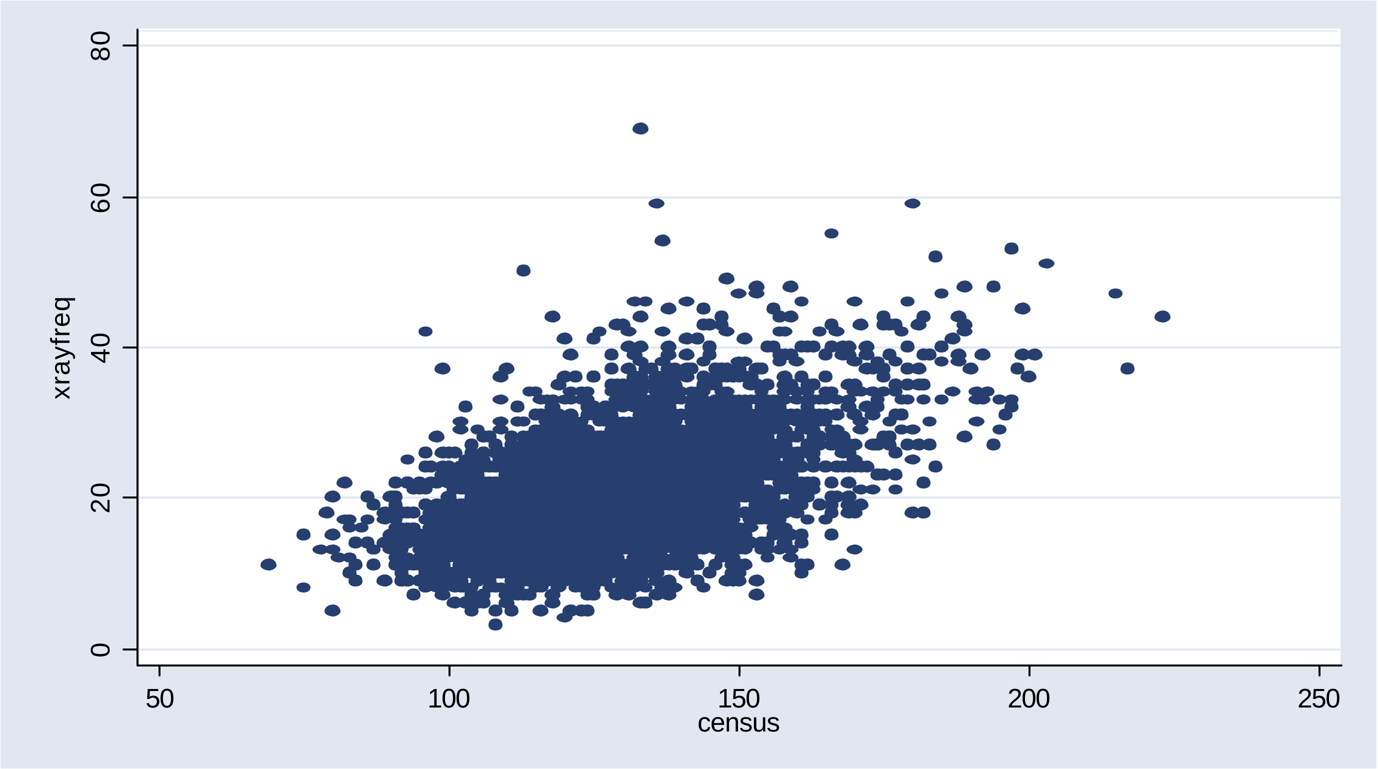


Series: census
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5
frequency



Series: xrayfreq
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5
frequency

Continue…



So..

- Residuals are

$$R(t) = Y(t) - \beta_0 - \beta_1 X(t) - S_1(t) - S_2(t) - \ldots - S_6(t)$$

- Fit arma(2,1) model to residuals
- Dynamically estimate the coefficients

Dynamic Model

$$Y_t = E^T ?_t$$

$$?_t^T = (m_t, x_t, x_{t-1})$$

$$\mathbf{E}^T = (1, 1, 0)$$

$$?_t = \mathbf{G} ?_{t-1} + \mathbf{w}_t$$

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f_1 & f_2 \\ 0 & 1 & 0 \end{pmatrix}$$

Forward Filtering

Posterior at t-1, $[?_{t-1} \mid D_{t-1}] \sim N(\mathbf{m}_{t-1}, \mathbf{C}_{t-1})$

Prior at t, $[?_{t} \mid D_{t-1}] \sim N(\mathbf{a}_{t}, \mathbf{R}_{t})$

$$\mathbf{a}_t = \mathbf{G}_t \mathbf{m}_{t-1}, \ \textit{and} \ \ \mathbf{R}_t = \mathbf{G}_\mathbf{t} \mathbf{C}_{\mathbf{t-1}} \mathbf{G}_\mathbf{t}^\mathbf{T} + \mathbf{W}$$

Posterior at t, $[?_{t} \mid D_{t}] \sim N(\mathbf{m}_{t}, \mathbf{C}_{t})$

$$\mathbf{m}_\mathbf{t} = \mathbf{a}_\mathbf{t} + \mathbf{A}_\mathbf{t} \mathbf{e}_\mathbf{t} \ \ \textit{and} \ \ \mathbf{C}_\mathbf{t} = \mathbf{R}_\mathbf{t} - \mathbf{A}_\mathbf{t} \mathbf{Q}_\mathbf{t} \mathbf{A}_\mathbf{t}^\mathbf{T}$$

$$\mathbf{A}_\mathbf{t} = \mathbf{R}_\mathbf{t} \mathbf{F}_\mathbf{t} \mathbf{Q}_\mathbf{t}^{-1} \ \ \textit{and} \ \ e_t = Y_t - f_t$$

$$f_t = \mathbf{F}_t^T \mathbf{a}_t \ \ \textit{and} \ \ \mathbf{Q}_t = \mathbf{F}_\mathbf{t}^\mathbf{T} \mathbf{R}_\mathbf{t} \mathbf{F}_\mathbf{t} + \mathbf{V}_\mathbf{t}$$

Backward sampling

- At t=n, sample $?_n$ from$(?_n | D_n) \sim N(\mathbf{m}_n, \mathbf{C}_n)$

$$for\ \ t = n-1,\ n-2, \ldots,\ 1,\ sample\ \ ?_t^{\ *}\ \ from\ \ P(?_t \mid D_t,\ ?_{t+1}^{\ *})$$

$$?_{t+1} = (m_{t+1}, x_{t+1}, x_t)$$

$$\downarrow$$

$$?_t = (m_t, x_t, x_{t-1})$$

- $Build\ \ \ up\ \ \Theta = (?_1^{\ *},\ ?_2^{\ *}, \ldots,\ ?_n^{\ *})$

Posterior distributions

$$\Lambda(\text{parameter set}) = \left(\mathbf{f}, \varphi, \mathrm{U}, \Gamma, \psi_0, \sigma_{0,} \rho_0\right)$$

$$L(\mathbf{z}_0, y_1, y_2, \ldots, y_t \mid \Lambda) = P(y_t \mid y_1, \ldots, y_{t-1}, \Lambda) \times P(y_{t-1} \mid y_1, \ldots, y_{t-2}, \Lambda)$$

$$\ldots \times P(y_2 \mid y_1, \mathbf{z}_0, \Lambda) \times P(y_1 \mid \mathbf{z}_0, \Lambda) \times P(\mathbf{z}_0 \mid \Lambda)$$

$$= P(y_t \mid y_{t-1}, y_{t-2}, \Lambda) \times P(y_{t-1} \mid y_{t-2}, y_{t-3}, \Lambda) \times$$

$$\ldots \times P(y_1 \mid \mathbf{z}_0, \Lambda) P(z_0 \mid \Lambda)$$

$$= P(\mathbf{z}_0 \mid \Lambda) \prod_{i=1}^{t} \exp\left(- \frac{(y_i - m_i - f_1 x_i - f_2 x_{i-1})^2}{2U(1 + j^2)} \right)$$

Posteriors…

$$P(\mathbf{f} \mid d, \Theta_t = (?_1, \ldots, ?_t), D_t) =$$

$$\propto P(\mathbf{f} \mid D_0) P(Z_0 \mid y_0, s_0, r_0, D_0) \prod_{i=1}^{t} \exp\left(-\frac{(y_t - \mathbf{f}^T ?_t)^2}{2U(1+j^2)} \right)$$

$$\propto P(\mathbf{f} \mid D_0) \prod_{i=1}^{t} \exp\left(-\frac{(y_t - m_t - f_1 x_t - f_2 x_{t-1})^2}{2U(1+j^2)} \right)$$

Continue…

$$f_1 \mid \cdot \sim N(k_{2.1}/k_{1.1}, 1/k_{1.1}), where$$

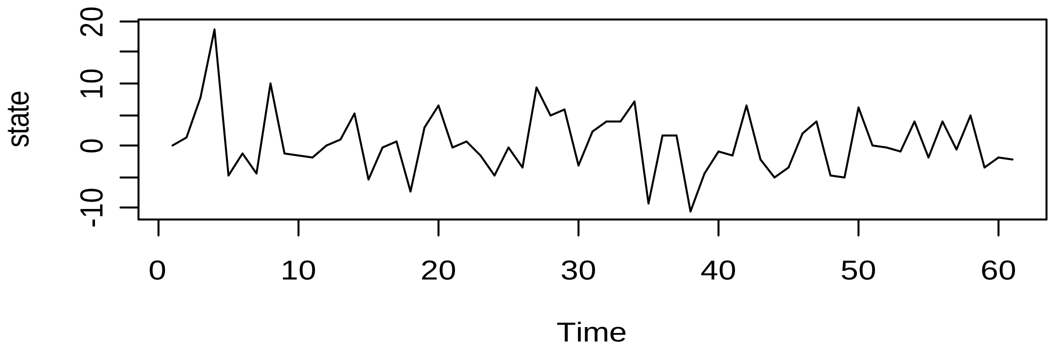
$$k_{1.1} = \frac{1}{s_{f_1}^{\;2}} + \sum_{i=1}^{t} \frac{(y_{i-1} - m_{i-1})^2}{U(1 + j^2)}$$

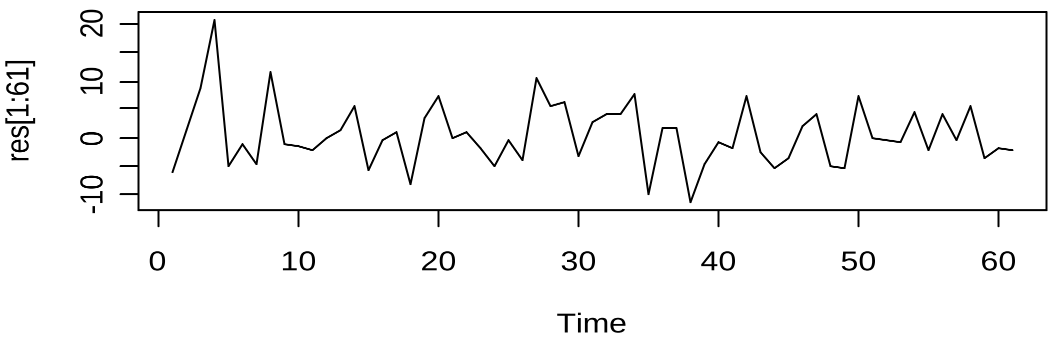
$$k_{2.1} = \frac{\mu_{\phi_1}}{\sigma_{\phi_1}^{\;2}} + \sum_{i=1}^{t} \frac{(y_{i-1} - \mu_{i-1})(y_i - \mu_i) - \phi_2 (y_{i-1} - \mu_{i-1})(y_{i-2} - \mu_{i-2})}{U(1 + \varphi^2)}$$

$$\phi_2 \mid \cdot \sim N(k_{2.2}/k_{1.2}, 1/k_{1.2}), \text{where}$$

RESULTS

4000 times iteration with 3000 burn-in





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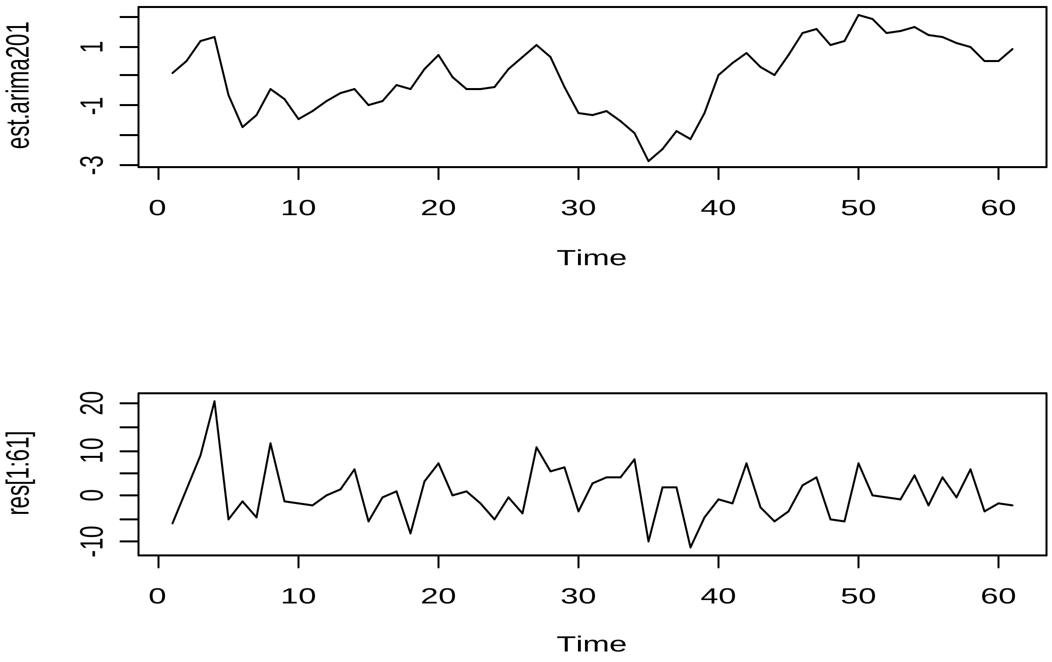
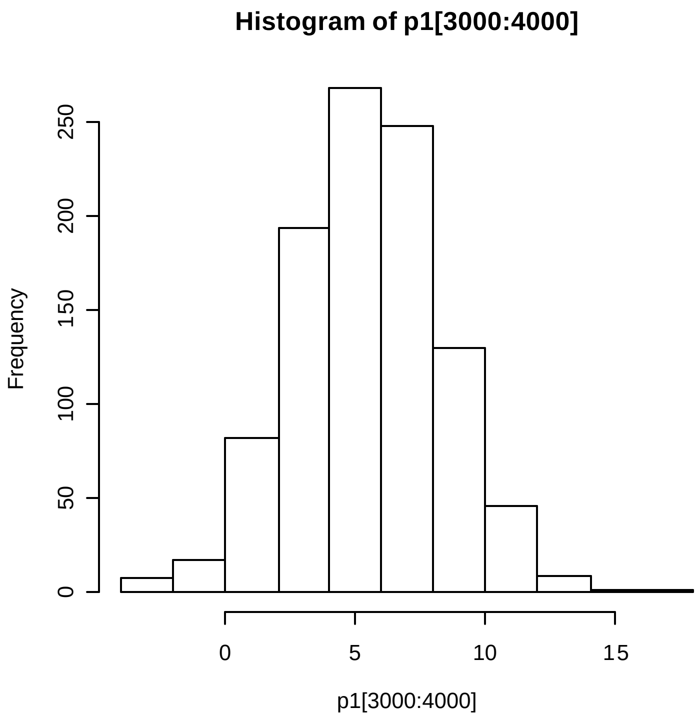
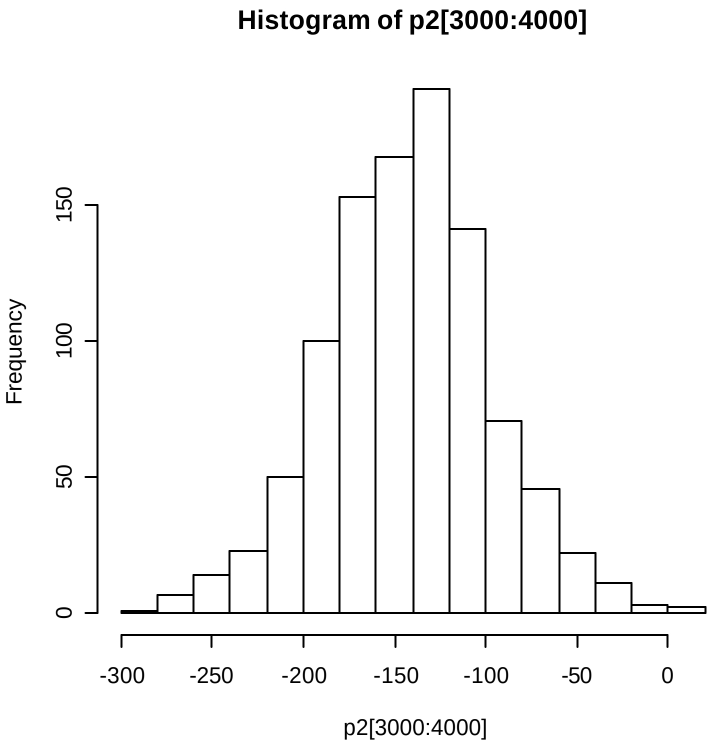


est.arima201
1
-1
-3
0
10
20
30
40
50
60
Time
res[1:61]
20
10
0
-10
0
10
20
30
40
50
60
Time


Histogram of p1[3000:4000]
Frequency
0
50
100
150
200
250
0
5
10
15
p1[3000:4000]


Histogram of p2[3000:4000]
Frequency
0
50
100
150
-300
-250
-200
-150
-100
-50
0
p2[3000:4000]

Further Study

- Discrete uniform prior on model order
- Informative priors on AR(P) to get stationarity
- Instead of using residuals, multivariate dynamic model for (total count, xray-count)