Bayesian Model Selection With Reversible Jump MCMC

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Reference

- Green, P, J. (1995), Reversible jump Markov chain Monte Carlo computation and Bayesian model determination, Biometrika 82: 711-732
- Andrieu, C, Freitas, JFG, and Doucet, A, Sequential Bayesian Estimation and Model Selection Applied to Neural Networks, CUED/F-INFENG/TR 341, Cambridge University, 1999.

Model Selection Overview

- Akaike information criterion (AIC)
- Minimum description length (MDL)
- Bayesian factor
- Forward/Backward

- Joint model selection

Joint Model Selection

- Problem statement
  - Model candidates: $\mathrm{M}_k, k \in \mathcal{K}, k \leqslant k_{max}$
  - Unknown parameters: $\theta^{(k)} \in \mathcal{R}^{n_k}$
  - Data observed: y
  - Joint density: $p(k, \theta^{(k)}, y) = p(k)p(\theta^{(k)} | k)p(y | k, \theta^{(k)}),$
  - Posterior $p(k, \theta^{(k)} | y) = p(k | y)p(\theta^{(k)} | k, y)$

Joint Model Selection

- Example 1 :

$$k \in \{1,\ \ 2\}$$

$$\mathrm{M}_1 :\quad q^{(1)} = q \in R$$

$$\mathrm{M}_2 :\quad q^{(2)} = (q_1, q_2) \in R^2$$

- Example 2 :

$$k = \{1,\ 2,\ \ \cdots,\ k_{\max}\}$$

$$\mathrm{M}_k : \begin{cases} x_t = \sum_{i=1}^{k} a_i x_{t-i} + s_{v,k} u_t \\ y_t = x_t + s_{w,k} w_t \end{cases}$$

- Example 3:

$$k = \{1,\ \ 2\}$$

$$\mathrm{M}_1 :\quad y \sim Gamma(a, b)$$

$$\mathrm{M}_2 :\quad y \sim Lognormal(m, s^2)$$

MCMC

- MCMC Computation
  - Gibbs sampling $\quad$ Accept $q_{\mathrm{i}}^{(\mathrm{k})} \sim p(\cdot \mid q_{\setminus i}^{(k)})$
  - M-H sampling $\quad$ Generate $\tilde{q} \sim q(\cdot \mid q)$

$$\text{Accept} \quad \min\left\{1, \frac{f(\tilde{q})q(q \mid \tilde{q})}{f(q)q(\tilde{q} \mid q)}\right\}$$

Reversible Jump MCMC

- Dimension Matching Transformation

$$(k, q^{(k)}) \to (k', q^{(k')}) \iff (k', q^{(k')}) \to (k, q^{(k)})$$

1. Model transformation probability $p(k' | k)$

2. $q^{(k')} = g_{1k \to k'}(q^{(k)}, U), \quad U \sim q_{k \to k'}(k, \cdot)$

3. $q^{(k)} = g_{1k' \to k}(q^{(k')}, U'), \quad U' \sim q_{k' \to k}(k', \cdot)$

4. $(q^{(k')}, U') = g_{k \to k'}(q^{(k)}, U) = (g_{1k \to k'}(q^{(k)}, U), g_{2k \to k'}(q^{(k)}, U))$

5. $(q^{(k)}, U) = g^{-1}_{k \to k}(q^{(k')}, U')$

Reversible Jump MCMC

- Acceptance Probability:

$$\min\left\{1, \frac{p(k'\mid y)\,p_{k'|k}\,q_{k'\to k}\left(q^{(k')},u'\right)p_{k'}\left(q^{(k')}\mid y\right)}{p(k\mid y)\,p_{k|k'}\,q_{k\to k'}\left(q^{(k)},u\right)p_{k}\left(q^{(k)}\mid y\right)}\left|\det\frac{\partial g_{k\to k'}(q^{(k)},u)}{\partial\left(q^{(k)},u\right)}\right|\right\}$$

Dimension Matching Transformation

Example:

$$\left(q_1,\ q_2\right) = g_{1->2}(q, U) = \left(q + U,\ q - U\right)$$

$$(q,\ U) = g_{2->1}(q_1,\ q_2) = (\frac{1}{2}(q_1 + q_2),\ \frac{1}{2}(q_1 - q_2))$$

$$\det \frac{\partial g_{1->2}(q, U)}{\partial (q, U)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

Sequential Inference

- Using Sequential Monte Carlo (Particle)
  - Select Importance function
  - Calculate importance weight
  - Importance weight resampling
  - Reversible JUMP MCMC move

Curve fitting

Given $x_1, x_2, \cdots, x_n$

Observe $y_1, y_2, \cdots, y_n$

Use the following model to fit y

$$\mathbf{y}_t = \boldsymbol{b}_t + \boldsymbol{\beta}_t' \mathbf{x}_t + \mathbf{n}_t \qquad k_t = 0$$

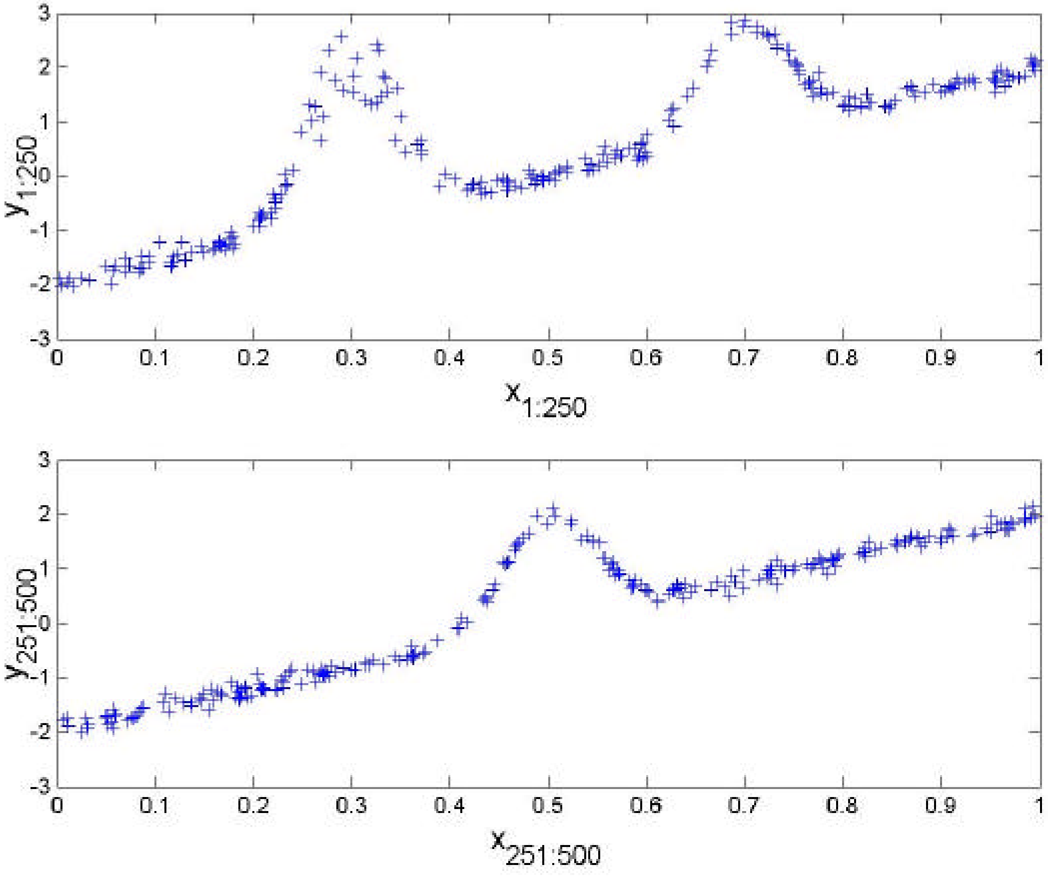
$$\mathbf{y}_t = \sum_{j=1}^{k_t} \mathbf{a}_{j,t} \phi_t(\|\mathbf{x}_t - \boldsymbol{\mu}_{j,t}\|) + \boldsymbol{b}_t + \boldsymbol{\beta}_t' \mathbf{x}_t + \mathbf{n}_t \qquad k_t \geq 1$$

Model Specification

- Set Base function to Gaussian
- Use absolute value as distance metric
- Data are generated from:

$$y_t = \begin{cases} 4x_t - 2 + \left(2 + \frac{t}{150}\right)\exp(-15(x_t - 0.3)^2) + 2\exp(-15(x_t - 0.7)^2) + n_t & 1 \le t \le 250 \\ 4x_t - 2 + 2\exp(-15(x_t - 0.5)^2) + n_t & 250 < t \le 500 \end{cases}$$

Model Data



3
2
1
0
-1
-2
-3
$y_{1:250}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
$x_{1:250}$
3
2
1
0
-1
-2
-3
$y_{251:500}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
$x_{251:500}$

Model Selection